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A geometric treatment of reduction of order of ordinary difference equations

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Abstract. We generalize the theory of Lie symmetries of ordinary difference equations to the non-autonomous case. A coordinate-invariant treatment in which solutions are sections of a fibre-bundle is employed. It is shown that Lie symmetries of difference equations must be projectable, which is not the case for differential equations. In fact this result extends to partial difference equations. We also show that a time-like symmetry can be used to reduce a non-autonomous difference equation to the autonomous case. Examples are given of this process and of the reduction of order of a non-autonomous system.

1. Introduction

With the increasing use of numerical techniques in mathematics and physics, considerable effort has been directed in recent times to the study of discrete systems. In particular, difference equations are of interest both as approximations to differential equations and as models of fundamentally discrete systems in biology, economics and in physics. In either case, knowledge of the symmetries of difference equations is of great importance. Symmetries can be used to reduce the order of a given system [6] or to produce complex integrable systems by reducing simpler, larger systems [3, 9], as is the case with differential equations. Where a difference equation is being used to generate numerical approximations to a given differential system, it is important to know if symmetries of the differential equations will be preserved accurately in the numerical results.

The reverse process may be more important. If we know the symmetries of a system of ordinary differential equations (ODEs), we can use them to reduce the order of the system *before* discretizing. On the other hand, if we have a discrete system with symmetries which we wish to replace with a continuous system, it is appropriate to reduce the order of the discrete system first. This paper explains how this can be done.

Surprisingly, the application of Lie symmetry techniques to difference equations is comparatively recent, seeming to originate in the work of Maeda [5, 6]. Considerable developments and generalizations have been made in this area, see for example [10, 12].

Returning to the work of Maeda [6], we find he restricts his attention to autonomous ordinary difference equations. The goal of this paper is to generalize his results to arbitrary ordinary difference equations (OΔEs). For ordinary differential equations the shift from autonomous to non-autonomous equations is straightforward, since the independent variable is real or complex and can therefore be treated as an extra dependent variable. However, in his treatment of OΔEs Maeda restricts the independent variable to take only discrete values, so if the non-autonomous generalization is not to be trivial this restriction must be removed.

Also, the vector field defined by a set of ODEs can be used to impose an equivalence relation on the generators of symmetries, such that every symmetry has an *evolutionary representative*. This is not the case with OΔEs, and this will be seen to be an important difference. In this paper the independent variable will always be considered as a coordinate on either $\mathbb{R}$ or $\mathbb{C}$, and the step (or mesh) size will be taken to be one. An nth-order system of OΔE then has the form

$$u^j(x+n) = f^j(x, u(x), \ldots, u(x+n-1)) \qquad \forall\, x \in \mathbb{R} \quad j = 1, \ldots, m.$$

Beyond this point we will consider only first-order equations explicitly. In many cases the treatment given will extend to higher-order equations, either by rewriting them as first-order equations of higher dimension or by the more sophisticated discrete jet-space approach used in [2].

The paper is organized as follows. In section 2 we describe the geometrical setting to be used and how difference equations appear from that perspective. Section 3 contains some necessary and sufficient conditions for the existence of a solution satisfying certain initial conditions. There are also definitions of linearizability and reduction of order. Symmetries of OΔEs are defined in section 4: in particular, we note differences that arise between autonomous and non-autonomous OΔEs, as well as between OΔEs and differential equations. For example, all symmetries of non-autonomous OΔEs must be projective. Following this we discuss evolutionary and time-like symmetries in section 5 and show that any system of first-order OΔEs can be transformed into an autonomous system. An example of this is given.

In section 6 we examine the relationship between symmetries and reduction of order. This includes the conditions under which symmetries pass to the reduced system to allow reduction of order by two or more. We also show how to re-construct a solution of the original system from a solution of the reduced system and the extra complexity due to time dependence. Again examples are given.

2. Difference equations

It is fruitful to study difference equations in a coordinate-independent setting, where a transformation can be interpreted as choosing new coordinates to describe the same geometrical object. Thus we consider the product space $E \simeq \mathbb{R} \times M$, where M is a C^∞ (smooth), m-dimensional real manifold. To deal in a satisfactory way with simplifying coordinate systems on E, we will use the notion of a trivialization of E, given by

$$\Theta : E \to \mathbb{R} \times M$$

$$\Theta : p \mapsto (\pi(p), \theta(p)).$$

Note that π is the projection to the first component. The set $M_x := \pi^{-1}(x)$, $x \in \mathbb{R}$ will be called the fibre over x and is diffeomorphic via the restriction $\theta_x := \theta|_{M_x}$ to M. In other words, θ_x defines time-dependent coordinates on M_x. We will use u to represent a point in M and x a point in $\mathbb{R}$, so that $\Theta(p) = (x, u) \in \mathbb{R} \times M$. Given a particular choice of local coordinates for M, the coordinates of u will typically be written as $(u^1, \ldots, u^m)$.

If M and N are manifolds and $f : M \to N$ is a smooth map, then the tangent space of M at u will be written $T_u M$ and the derivative map at u of f is $f_*(u) : T_u M \to T_{f(u)} N$.

A section of E is a map $\rho : \mathbb{R} \to E$ which satisfies $\pi \circ \rho = \mathrm{id}_{\mathbb{R}}$.

The initiated will recognize that the constructions of this paper could easily be generalized to the case of a fibre bundle with typical fibre M, arbitrary one-dimensional base and projection π.

Definition 1. Let $\Phi : E \to E$ be a smooth map such that

$$\pi \circ \Phi(p) = \pi(p) + 1 \qquad p \in E$$

and $\rho : \mathbb{R} \to E$ a smooth section. Then ρ is a solution of the first-order OΔE defined by Φ iff

$$\rho(x+1) = \Phi \circ \rho(x) \qquad \forall\, x \in \mathbb{R}.$$

The image of ρ is a submanifold of E which can also be viewed as the graph of a function f defined by $\rho(x) =: (x, f(x))$. The solution submanifold is then the graph

$$\{(x, u) \in \mathbb{R} \times M : u = f(x)\}$$

leading to the common notation

$$u^j(x+1) = f^j(x, u(x)) \qquad j = 1, \ldots, m.$$

The danger of such notation is that when we define a vector field X on $\mathbb{R} \times M$, $X(x, u)$ depends on the point (x, u) but *not* on which of infinitely many solutions ρ satisfying $\rho(x) = u$ is being considered. If u is used for both maps $\mathbb{R} \to M$ and points in M, confusion is difficult to avoid.

3. Initial conditions and solutions

There is a significant extra richness introduced by requiring the independent variable to be continuous. Restricting ourselves to the first-order case for clarity, the mapping $\Phi : E \to E$ naturally induces a mapping on the space of sections $[0, 1) \to E$. For the sake of definiteness, let us consider only smooth functions (although the restriction is not critical), denoted $\Gamma([0, 1), E)$. By a slight abuse of notation, let Φ also stand for the induced map

$$\Phi : \Gamma([0, 1), E) \longrightarrow \Gamma([1, 2), E).$$

Hence we expect that the initial condition for a first-order OΔE is an element of $\Gamma([0, 1), E)$. If we require that the corresponding solution be smooth, it is necessary to impose compatibility conditions.

Proposition 1. Suppose $\rho_i \in \Gamma([0, 1), E)$ and let $\Phi : E \to E$ define a first-order OΔE. Then there is a smooth section $\rho : \mathbb{R}^+ \to E$ such that

$$\rho(x+1) = \Phi \circ \rho(x) \qquad \forall\, x \in \mathbb{R}^+$$

$$\rho(x) = \rho_i(x) \qquad \forall\, x \in [0, 1)$$

if and only if ρ_i satisfies

$$\left(\frac{\mathrm{d}\rho_i}{\mathrm{d}x}\right)^k (1) = \left(\frac{\mathrm{d}(\phi \circ \rho_i)}{\mathrm{d}x}\right)^k (0) \qquad k = 0, 1, 2, \ldots$$

where the derivatives are to be understood as left- or right-derivatives and limits as appropriate.

An important consequence of this result is that for any given first-order OΔE, the set of solutions does not have bounded derivatives. By comparison, a first-order ODE can be interpreted as a vector field, so that derivatives of solutions are determined by the ODE rather than the initial conditions. The derivatives of solutions to ODEs are therefore bounded for smooth, non-singular ODEs on compact domains. This difference will have an important consequence when we discuss symmetries of OΔEs in the next section.

More immediate is the problem of knowing what it means to 'solve' an OΔE. We know that for a non-singular, first-order ODE given by a vector field V on E which projects to a unit vector on the first component, $\pi_* V = \partial/\partial x$, the set of all (local) solutions is parametrized by any surface transverse to V, in particular by M. The flow of the vector field V carries forward the coordinates on $M \simeq \pi^{-1}(x_0)$ so that in some neighbourhood of $\pi^{-1}(x_0)$ there is a trivialization θ such that $\theta_* V = 0$. Hence there are coordinates in which $V = \partial/\partial x$, corresponding to the trivial ODE $\dot{u} = 0$. At this stage we can say that the ODE has been completely solved, since the integral curves are simply $u = u_0$. While carrying out this procedure may not be possible in practice, we know that such a complete solution exists.

The situation is changed significantly when working with difference equations. Conjugating Φ with the chosen trivialization defines a parametrized automorphism of M

$$\tilde{\Phi}_x := \theta_{x+1} \circ \Phi_x \circ \theta_x^{-1}.$$

(We will always assume that x and $x+1$ are within the domain of the same trivialization).

Definition 2. We say that a first-order OΔE given by $\Phi : E \to E$ is linearized on $U \subset E$ if there is a trivialization $\Theta : E \to \mathbb{R} \times M$ given in which

$$\tilde{\Phi}_x(u) = \mathrm{id}_M \qquad \Theta^{-1}(x, u) \in U.$$

If such a trivialization exists it is called linearizing.

Now suppose there exists a linearizing trivialization and that M is orientable. The Jacobian of a diffeomorphism $M \to M$ can then be defined and its sign is independent of coordinates. If $g : \mathbb{R} \times M \to M$ defines a change of trivialization by

$$\theta_x \mapsto g_x \circ \theta_x \qquad x \in \mathbb{R}$$

then the resulting transformation of $\tilde{\Phi}$ is

$$\tilde{\Phi}_x \mapsto g_{x+1} \circ \tilde{\Phi}_x \circ g_x^{-1}.$$

Now since g_x is smooth and non-singular for all $x \in \mathbb{R}$, its Jacobian $J(g_x)$ is non-zero and continuous on $\mathbb{R}$. Consequently $J(g_{x+1})/J(g_x) > 0$ and the sign of $J(\tilde{\Phi}_x)$ is invariant under changes of trivialization, so if Φ can be linearized on U then

$$J(\tilde{\Phi}_x) > 0 \qquad \forall\ (x, \theta^{-1}(u)) \in U.$$

As an example recall the logistic map, a first-order, one-dimensional autonomous OΔE given in the standard coordinates by

$$\tilde{\Phi}_x(u) = au(1-u) \qquad \forall\ x \in \mathbb{R}$$

which is known to be chaotic if $a = 4$. Thus $J(\tilde{\Phi}_x)(u) = a(1-2u)$, which has a zero when $u = \frac{1}{2}$. Thus the logistic equation cannot be linearized on any open neighbourhood having non-empty intersection with $\theta^{-1}(\frac{1}{2})$. In fact we can immediately prove a stronger result: we find that an OΔE has a solution in this strong sense iff the OΔE is a time-one integral of an ODE.

Proposition 2. Let M be a smooth, real manifold of dimension m and let $\Phi : E \to E$ define a non-autonomous difference equation. Then Φ can be linearized on $U \subset E$ if and only if there is a smooth, complete vector field V with $\pi_* V \neq 0$ on U such that $\Phi(p)$, $p \in U$, is the unique point on the integral curve of V through p which projects by π to $\pi(p) + 1$.

Proof. Suppose first that Φ can be linearized, with linearizing trivialization $\Theta := \pi \times \theta$. In the coordinates defined on E by Θ (together with any atlas on M) it can be seen that $V := \frac{\partial}{\partial x}$ has the appropriate property.

To prove the converse, note that the flow of a complete non-singular vector field V such that $\pi_* V \neq 0$ on a fibre-bundle π defines a trivialization $\Theta := \pi \times \theta$, with the property $\theta_* V = 0$. Thus if $\rho : \mathbb{R} \to E$ is the integral section of V through $p \in E$, $\theta \circ \rho(x) = \theta(p)$ for all $x \in \mathbb{R}$, from which it follows that $\tilde{\Phi}_x = \mathrm{id}_M$ for all $x \in \mathbb{R}$. □

Theorem 3. Suppose that $\pi : E \to \mathbb{R}$ has typical fibre M which is orientable. Let $\Phi : E \to E$ define a first-order OΔE such that $\tilde{\Phi}$ has positive Jacobian. Then Φ can be linearized on a tubular neighbourhood of any solution. In the case where the typical fibre M of E is homotopic to $\mathbb{R}^m$, Φ can be linearized on E.

Proof. Let $\rho : \mathbb{R} \to E$ be a section of $\pi : E \to \mathbb{R}$ such that $\Phi \circ \rho(x) = \rho(x+1)$, $x \in \mathbb{R}$, so that ρ is a solution of the OΔE.

Now define a Riemannian metric g_1 on M_1. From g_1 we can determine an exponential map

$$\exp : T_{\rho(1)} M_1 \to U_1$$

where $U_1 \subseteq M_1$, $\rho(1) \in U_1$. Note that if $M \simeq \mathbb{R}^m$ then g_1 can be chosen to be the Euclidean metric and $U_1 = M_1$.

The pull-back of g_1 by Φ induces a metric $g_0 := \Phi^* g_1$ on M_0, with the property that geodesics of g_0 map to geodesics of g_1 under Φ. Now g_1 can be smoothly extended to g_{x+1}, $|x| < 1/4$, with Φ used to define g_x. Since g is Riemannian, g_x for $1/4 \leqslant x \leqslant 3/4$ can be defined by extending from each side and using a smooth partition of unity on the overlap.

We have now defined g on $\pi^{-1}(-\frac{1}{4}, \frac{5}{4})$ in such a way that $g_x = \Phi^* g_{x+1}$, $|x| < 1/4$, so g can be extended with this property to the whole of E by iterating pull-backs by Φ and Φ^{-1}. Thus geodesics of g which are wholly contained within a fibre M_x map to geodesics on a fibre M_{x+1}, $x \in \mathbb{R}$ and the diagram

$$\begin{array}{ccc} T_{\rho(x)} M_x & \xrightarrow{\Phi_*} & T_{\rho(x+1)} M_{x+1} \\ \downarrow \exp & & \downarrow \exp \\ U_x & \xrightarrow{\Phi} & U_{x+1} \end{array}$$

with $\Phi(U_x) \subseteq U_{x+1}$, commutes.

Let $\Theta = \pi \times \theta : E \to \mathbb{R} \times M$ be a trivialization of E. Composing θ with the inverse of the exponential map we obtain a new trivialization

$$\theta' := \theta \circ \exp^{-1}.$$

Let $\tilde{\Phi}'$ and $\tilde{\Phi}$ be the automorphisms of M corresponding to θ' and θ, respectively. Then $\tilde{\Phi}'_x = \mathrm{id}$ on an open neighbourhood $\theta(U_x)$ of $\theta \circ \rho(x)$ if and only if $\Phi_{x*} = \mathrm{id}_{T_{\theta \circ \rho(x)} M}$, i.e.

iff Φ_{x*} is the identity at a single point. Therefore θ' will be a linearizing transformation for Φ on a tubular neighbourhood of the image of ρ iff we can choose θ_* to be linearizing along ρ. This in turn can be done iff there is a change of trivialization $g : \mathbb{R} \times M \to M$ with derivative at $\theta \circ \rho(x)$ G_x such that

$$G_{x+1} = G_x \tilde{\Phi}_{x*}^{-1}.$$

This last equation is just a non-autonomous OΔE on the Lie group $GL(m, \mathbb{R})$, which always has a smooth solution provided that $\tilde{\Phi}_{x*}^{-1}$ maps the identity connected component to itself. This is the case provided that the Jacobian of $\tilde{\Phi}_{x*}^{-1}$, hence of $\tilde{\Phi}_{x*}$, is positive. □

The following corollary follows immediately from the theorem.

Corollary 4. If $\Phi : E \to E$ defines a first-order OΔE, then the OΔE defined by the iterate Φ^2 can always be linearized and is therefore the integral of a differential equation in the sense of proposition 2.

3.1. Reduction of order

If an OΔE does not have a complete linearization, or if it cannot be found, it may still be possible to reduce it to an OΔE of lower dimension.

Let $\pi_F : F \to \mathbb{R}$ be a (trivial) fibre bundle with connected typical fibre N, $\dim N = n < m$, such that there is a projection $\mu : E \to F$ which makes the diagram

$$\begin{array}{ccc} E & \xrightarrow{\mu} & F \\ \downarrow \pi & & \downarrow \pi_F \\ \mathbb{R} & \xrightarrow{\mathrm{id}} & \mathbb{R} \end{array}$$

commute.

Definition 3. If there is a map $\phi : F \to F$ such that

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E \\ \downarrow \mu & & \downarrow \mu \\ F & \xrightarrow{\phi} & F \end{array}$$

commutes, then we say that there is a reduction of $\Phi : E \to E$ to $\phi : F \to F$, which defines a first-order OΔE of dimension n.

The above situation can be described in terms of local coordinates which are adapted to the projection μ. An atlas for E is said to be adapted to μ if in each local chart the coordinates are of the form $(x, u^1, \ldots, u^n, w^{n+1}, \ldots, w^m)$, where the first coordinate is constant on each fibre $\pi^{-1}(x)$ and the next n, $u^1, \ldots, u^n$ are constant on each fibre of μ, $\mu^{-1}(q)$, $q \in F$.

Proposition 5. There is a reduction of Φ to ϕ iff and only if there are coordinates adapted to μ such that

$$\frac{\partial}{\partial u^j} \Phi^k = 0 \qquad j = 1, \ldots, n \quad k = n+1, \ldots, m.$$

Proof. We prove the forward implication first. Let X be a vector field satisfying

$$\pi_* X = \mu_* X = 0$$

so that it is tangent to the fibres of μ and of π. Then at each point of E, $X \in \operatorname{span}\{\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}\}$. Now if σ_t is the flow of X, $\mu \circ \sigma_t(p) = \mu(p)$ for all $p \in E$ and t in some neighbourhood of zero. Assuming Φ to be reducible,

$$\mu \circ \Phi \circ \sigma_t(p) = \phi \circ \mu \circ \sigma_t(p) = \phi \circ \mu(p) = \mu \circ \Phi(p).$$

Differentiating w.r.t. t at zero we obtain

$$\mu_* \Phi_* X = 0.$$

However, the kernel of μ_* is spanned by $\{\frac{\partial}{\partial w^{n+1}}, \ldots, \frac{\partial}{\partial w^m}\}$, which proves the result.

Conversely, we have $\mu_* \Phi_* X = 0$ for all such X, which can be true only if the value of $\mu \circ \Phi$ is independent of $u^1, \ldots, u^n$. Thus we can define $\phi \circ \mu := \mu \circ \Phi$, satisfying definition 3. □

It should be noted that if $\dim N = 0$, so that $F = \mathbb{R}$, then Φ is linearized in the adapted coordinates.

4. Symmetries

Definition 4. Let X be a smooth vector field on E, generating a flow $\psi_\tau = \exp(\tau X)$. We say that X generates a symmetry (or simply is a symmetry) of a first-order OΔE defined by $\Phi : E \to E$ if the image under ψ_τ of each solution $\rho : \mathbb{R} \to E$ remains a solution for τ in some neighbourhood of zero: that is there exists $\epsilon > 0$ such that for all $\tau \in [-\epsilon, \epsilon]$,

$$\psi_\tau \circ \Phi = \Phi \circ \psi_\tau.$$

If we differentiate the above equation w.r.t. τ at $\tau = 0$, we obtain a necessary condition for X:

$$X \circ \Phi = \Phi_* X. \tag{1}$$

In the case of autonomous OΔEs we can use the exponential map to construct ψ_τ from X, to prove that the converse also holds and (1) can be taken as the definition of a symmetry vector field. However, there is an additional complication with non-autonomous OΔEs: an arbitrary flow ψ_τ may carry solutions to curves which are not the solution of *any* difference equation.

Lemma 6. Let $\Phi : E \to E$ define an first-order OΔE and let X be a smooth vector field on E with coordinate expression

$$\Theta_* X(x, u) = A(x, u)\frac{\partial}{\partial x} + B^j(x, u)\frac{\partial}{\partial u^j}$$

and flow

$$\exp(\tau X) =: \psi_\tau : E \longrightarrow E.$$

Then if for some j and some $(x, u) \in \mathbb{R} \times M$ we have $\partial A(x, u)/\partial u^j \neq 0$, there exists for all $\tau > 0$ a solution $\rho : \mathbb{R} \to E$ of Φ with image G_ρ such that $\psi_\tau G_\rho$ is not a section of $\pi : E \to \mathbb{R}$.

Proof. We may as well assume that $\partial A(0,0)/\partial u^j > 0$ and hence that $\partial A/\partial u^j > 0$ on some neighbourhood of $(0,0)$. In order for a curve γ in E to be the image of a section of $\pi : E \to \mathbb{R}$, there must be a parametrization $\gamma : \mathbb{R} \to E$ such that composition with the projection gives the identity map: $\pi \circ \gamma = \mathrm{id}_{\mathbb{R}}$, which in turn is possible only if the derivative of the composition is strictly positive: $\pi_*(\dot{\gamma}) > 0$.

Now consider a curve γ in E given in coordinates by

$$\Theta \circ \gamma : t \mapsto (t, 0, \ldots, 0, Kt, 0, \ldots, 0) \qquad t \in (-\epsilon, \epsilon)$$

where the $j+1$th entry is non-zero, K is constant and $\epsilon > 0$. Since $\pi \circ \gamma = \mathrm{id}_{\mathbb{R}}$ it is the image of a section. Provided $\epsilon < \frac{1}{2}$, γ can be smoothly extended to an allowable initial condition for Φ and hence to a solution.

The action of ψ_τ on γ gives

$$\begin{aligned}\Theta \circ \psi_\tau \circ \gamma : t \mapsto (t + \tau A \circ \gamma(t), \tau B^1 \circ \gamma(t), \ldots, Kt + \tau B^j \circ \gamma(t), \ldots, \ldots, \\ \tau B^m \circ \gamma(t)) + \mathrm{O}(\tau^2).\end{aligned}$$

Taking the first component and differentiating we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\pi \circ \psi_\tau \circ \gamma)(0) = 1 + \tau\left(\frac{\partial A}{\partial x} + K\frac{\partial A}{\partial u^j}\right) + \mathrm{O}(\tau^2)$$

so if we fix $\tau > 0$ and choose

$$K < -\left(\frac{1}{\tau} + \frac{\partial A}{\partial x}\right) \Big/ \frac{\partial A}{\partial u^j}$$

then $\pi_* \psi_{\tau *} \dot{\gamma}(0) < 0$. Thus $\psi_\tau G_{\rho_1}$ cannot be the image of any section of $\pi : \mathbb{R} \to E$. □

Theorem 7. If a vector field X on E generates a symmetry of an OΔE, then it is fibre preserving: there is a smooth function $A : \mathbb{R} \to \mathbb{R}$ such that

$$\pi_* X = A \frac{\partial}{\partial x}.$$

This implies that X has local coordinate expression

$$\Theta_* X(x, u) = A(x)\frac{\partial}{\partial x} + B^j(x, u)\frac{\partial}{\partial u^j}.$$

Moreover A is unit-periodic, $A(x+1) = A(x)$ for all $x \in \mathbb{R}$.

Proof. From definition 4 and the lemma, $\pi_* X(p)$ depends only on $\pi(p)$, so A exists. Since X generates a symmetry,

$$(A \circ \pi)\frac{\partial}{\partial x} = \pi_* \Phi_* X = \pi_*(X \circ \Phi) = A \circ \pi \circ \Phi \frac{\partial}{\partial x}.$$

However, as $\pi \circ \Phi(p) = \pi(p) + 1$, this reduces to $A(x+1) = A(x)$ for all $x \in \mathbb{R}$. □

Note that this result extends to *partial* difference equations by simply restricting the base space to a one-dimensional submanifold.

This extra condition of projectability is sufficient when combined with the derivative condition (1) to ensure that a vector field generates a symmetry.

Theorem 8. Let $\Phi : E \to E$ define an OΔE. The vector field X on E generates a symmetry of Φ if and only if X satisfies the conditions

(i) $\Phi_* X = X \circ \Phi$;
(ii) $\pi_* X = A \frac{\partial}{\partial x}$, $A \in C^\infty(\mathbb{R})$.

Proof. The necessity has already been shown. Suppose ρ is a section of $\pi : E \to \mathbb{R}$ which solves the OΔE, that is $(\Phi \circ \rho)(x) = \rho(x+1)$. Since ρ is a section, its image intersects each fibre M_x in a single point. Now the second condition of the theorem implies that the flow of X maps fibres to fibres, so the image of $\psi_\tau \circ \rho$ also intersects each fibre in a single point and $\psi_\tau \circ \rho$ is therefore a section of π.

The first condition implies that $\psi_\tau \circ \Phi = \Phi \circ \psi_\tau$ for τ in some neighbourhood of zero, so solution sections map to solution sections. □

4.1. Algebraic properties of symmetries

It is known that the symmetry vector fields of an ODE on E form a Lie sub-algebra of the algebra of smooth vector fields on E, since the condition for X to be a symmetry of the ODE determined by the vector field V is just $[X, V] = \lambda V$. Below we investigate the algebraic properties of the symmetries of an OΔE.

Proposition 9. Let f, g be unit-periodic, real valued functions on $\mathbb{R}$. Then if X, Y are symmetry vector fields of Φ, so is $(\pi^* f)X + (\pi^* g)Y$. Here the pull-back $\pi^* f$ is defined as usual by

$$\pi^* f(p) = f \circ \pi(p) \qquad p \in E.$$

Proof. First, if X is fibre preserving then so is $(\pi^* f)X$ for any $f : \mathbb{R} \to \mathbb{R}$. If h is any function on E, then $\Phi_*(hX) = h\Phi_* X$, so we need to show only that $(\pi^* f) \circ \Phi = \pi^* f$. However, for all $p \in E$, $\pi \circ \Phi(p) = \pi(p) + 1$ and f is unit periodic. □

Hence the symmetry vector fields of Φ form a module over the ring of unit-periodic functions. It is also a Lie algebra over $\mathbb{R}$ with the usual commutator bracket:

Proposition 10. If X and Y are symmetry vector fields of an OΔE defined by Φ, then so is $[X, Y]$. Consequently the symmetry vector fields of Φ form an $\mathbb{R}$-Lie algebra.

Proof. In the case of autonomous OΔEs and autonomous symmetries this was shown by Maeda [5], but in any case the proof is trivial:

$$\Phi_*[X, Y] = [\Phi_* X, \Phi_* Y] = [X \circ \Phi, Y \circ \Phi] = [X, Y] \circ \Phi.$$

Also $\pi_*[X, Y] = [\pi_* X, \pi_* Y]$ is a well-defined vector field on $\mathbb{R}$, so $[X, Y]$ is fibre-preserving. Closure under addition and scalar multiplication is a special case of proposition 9. □

However, the periodic functions are not constants with respect to the the Lie derivation, so without further restriction it cannot be treated as a Lie algebra over the ring. As an $\mathbb{R}$-algebra, a consequence of proposition 9 is that it cannot be be finite dimensional.

5. Evolutionary and time-like symmetries

In the case of ODEs, fibre preserving symmetries (sometimes called *projectable* [7]) are not the only type. However, for first-order ODEs one can define *trivial symmetries*, which leave each solution curve invariant up to re-parametrization. Every symmetry vector field is equivalent, modulo a trivial symmetry, to a symmetry vector field with $\pi_* X = 0$ (and are therefore fibre-preserving). Such symmetries are called evolutionary and the definition is also important for non-autonomous OΔEs.

Definition 5. Let X generate a symmetry of the OΔE defined by $\Phi : E \to E$. We say that X is an evolutionary symmetry vector field if $\pi_* X = 0$, or in coordinates

$$X(x, u) = B^j(x, u)\frac{\partial}{\partial u^j}.$$

If X is evolutionary and $f \in C^\infty(\mathbb{R})$ then $X(\pi^* f) = 0$. Let the ring of smooth unit periodic functions on $\mathbb{R}$ be denoted by P.

Proposition 11. The set of evolutionary symmetry vector fields of an OΔE is a P-Lie algebra with the commutator as Lie product.

Proof. Since the fibres of π are integrable it is certainly an $\mathbb{R}$-Lie algebra. We have seen in proposition 9 that it is closed under multiplication by elements of $\pi^* P$, and if X and Y are evolutionary and $f, g \in P$ then

$$[(\pi^* f)X, (\pi^* g)Y] = \pi^*(fg)[X, Y].$$

□

In fact, one finds that theorems on symmetries of ODEs translate readily to theorems on OΔEs provided that attention is restricted to evolutionary symmetries.

Proposition 12. Let $\Phi : E \to E$ define a first-order OΔE which can be linearized on some open, simply connected domain $U \subset E$. Then there exist $m = \dim M$ point-wise independent, commuting, evolutionary symmetry vector fields for Φ on U.

Proof. From proposition 2 we know there exists a vector field V on U with flow Ψ_τ such that $\Psi_1 = \Phi$. Since U is a domain we can choose coordinates in which V is represented by $\frac{\partial}{\partial x}$ and evolutionary vector fields $X_j = \partial/\partial u^j$, $j = 1, \ldots, m$ using local fibre coordinates u^j. Being coordinate vector fields these will automatically satisfy the conditions

$$[V, X_j] = 0 \qquad j = 1, \ldots, m.$$

Now

$$0 = [V, X_j] = \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} \left(\Phi^{-1}_{\tau_0+\tau *} X_j \circ \Phi^{-1}_{\tau_0+\tau} - X_j\right) \qquad \tau_0 \in [0, 1].$$

Since $\Phi_0 = \mathrm{id}$ we have

$$\Phi^{-1}_{\tau *} X_j \circ \Phi^{-1}_{\tau} - X_j = 0 \qquad \tau \in [0, 1]$$

and setting $\tau = 1$ and multiplying by Φ_* we obtain

$$X_j \circ \Phi = \Phi_* X_j \qquad j = 1, \ldots, m$$

as required. $\square$

Proposition 13. Suppose that $\Phi : E \to E$ can be reduced to $\phi : F \to F$ on an open, simply connected domain U, as in definition 3. Then there exist on U $m - n$ point-wise independent, commuting, evolutionary symmetry vector fields for Φ.

Proof. Consider the local coordinates (x, u, w) of proposition 5. Proposition 12 can be applied to the restriction of Φ to the w coordinates on each chart. $\square$

For a first-order ODE, trivial symmetries can be characterized as those which leave each solution invariant up to re-parametrization, whereas non-trivial symmetries permute distinct solutions. In fact the generator of a trivial solution is a scalar multiple of the vector field which defines the ODE. Since this vector field has a nowhere zero projection, it is elementary linear algebra to show that any other symmetry generator can be made evolutionary by adding a suitable multiple of the trivial symmetry. Unfortunately, this method cannot be applied to OΔEs, since they cannot be defined by a single vector field.

Consider an arbitrary first-order OΔE defined by Φ and a solution $\rho_1 : \mathbb{R} \to E$. A vector field X generating a flow which leaves ρ_1 invariant up to reparametrization must be tangent to the graph of ρ_1. However, there always exists a second solution $\rho_2 : \mathbb{R} \to M$, $\rho_1 \neq \rho_2$ such that

$$\rho_1(k) = \rho_2(k) \qquad \rho_1'(k) \neq \rho_2'(k) \qquad k \in \mathbb{Z}.$$

Clearly if X is tangent to both ρ_j, then

$$X(k, \rho_1(k)) = 0 \qquad k \in \mathbb{Z}.$$

In fact ρ_2 could have been chosen to intersect ρ_1 an arbitrary, finite number of times in the interval $[0, 1)$, extending with the unit period over $\mathbb{R}$. So there is no non-singular symmetry vector field which leaves the graphs of both ρ_1 and ρ_2 invariant. Consequently, it is not possible to use the invariance of individual solutions as a definition of trivial symmetries for OΔEs. However, the following result demonstrates that time-like symmetries can substitute for trivial symmetries in some ways.

Definition 6. A vector field X on E is said to be time-like if for all $p \in E$ $\pi_* X_p \neq 0$.

The time-like vector fields are clearly the point-wise complement of the evolutionary ones.

Lemma 14. The OΔE defined by Φ is autonomous in a trivialization $\Theta = \pi \otimes \theta$ iff for any vector field Z on E,

$$Z \in \ker \theta_* \Rightarrow \Phi_* Z \in \ker \theta_*.$$

Proof. In coordinates, Φ is autonomous iff

$$\frac{\partial \Phi^j}{\partial x} = 0 \qquad j \neq 0$$

which is equivalent to $\Phi_* T = T$ for T the vector field satisfying $\theta_* T = 0$ and $\pi_* T = \frac{\partial}{\partial x}$. Suppose Φ is autonomous. Since θ_* has a one-dimensional kernel,

$$Z \in \ker\theta_* \Rightarrow Z = fT \qquad f \in C^\infty(E)$$

and therefore $\Phi_* Z = fT \in \ker\theta_*$.

Conversely, again using $\dim\ker\theta_* = 1$,

$$\theta_* Z = 0 \Rightarrow \theta_*\Phi_* Z = 0$$

implies that if $\theta_* Z = 0$ then

$$\Phi_* Z = fZ \qquad f \in C^\infty(E).$$

Differentiating the condition of definition 1 we have for any vector field Y on E that $\pi_*\Phi_* Y = \pi_* Y$, so

$$\pi_*\Phi_* Z = \pi_* Z = \pi_*(fZ) = f\pi_*(Z)$$

and $f = 1$. Now $\theta_x : M_x \to M$ is a diffeomorphism for each $x \in \mathbb{R}$, so the restriction of θ_* to a fibre M_x has maximal rank. Consequently if $Z \in \ker\theta_*$, $\pi_*(Z)$ is nowhere zero. Setting $T = Z/(\pi_* Z, \mathrm{d}x)$, with $(\cdot,\cdot)$ the standard inner product on $\mathbb{R}$, T satisfies $\theta_* T = 0$, $\pi_* T = \frac{\partial}{\partial x}$ and $\Phi_* T = T$. Hence Φ is autonomous with respect to the trivialization Θ. □

Theorem 15. There exists a trivialization of E in which a given OΔE defined by $\Phi : E \to E$ becomes autonomous on $U \subset E$ iff there is a complete vector field T on U with the properties

(i) $\pi_* T \neq 0$;
(ii) $\Phi_* T = T \circ \Phi$.

That is iff T is a time-like symmetry vector field of Φ.

Proof. Suppose first that Φ is autonomous in some trivialization $\Theta : E \to \mathbb{R} \times M$. Set T to be the unique vector field such that $\theta_* T = 0$ and $\pi_* T = \frac{\partial}{\partial x}$. From the lemma we have that $\Phi_* T = T$ and $T \circ \Phi = T$, so T is a symmetry vector field.

Now suppose T is given. From theorem 7, we know there is a unit-periodic function A on $\mathbb{R}$ such that $\pi_* T = A\frac{\partial}{\partial x}$, with $A \neq 0$ on U by hypothesis. Thus $1/A$ is a well defined unit-periodic function on U, so $\tilde{T} := \frac{1}{A}T$ is a symmetry vector field of Φ (by proposition 9) with $\pi_*\tilde{T} = \frac{\partial}{\partial x}$. We can then use $\tilde{T}$ to define a trivialization Θ of U such that $\theta_*\tilde{T} = 0$. However, θ_* has a one-dimensional kernel, so if $Z \in \ker\theta_*$ then $Z = f\tilde{T} = gT$, $f, g \in C^\infty(E)$. Since T is assumed to be a symmetry vector field,

$$\Phi_* Z = g\Phi_* T = g(T \circ \Phi) \in \ker\Phi_*.$$

Thus $\theta_* Z = 0$ implies $\theta_*\Phi_* Z = 0$. □

Theorem 16. For any first-order OΔE defined by $\Phi : E \to E$ there is a symmetry vector field T for Φ such that $\pi_* T \neq 0$ on E. Conseqeuently there is a trivialization for Φ in which it becomes autonomous.

Proof. Since the general linear group has only two connected components it is always possible for some $\epsilon > 0$ (by constructing a smooth homotopy with x as parameter), to choose a trivialization Θ so that

$$\tilde{\Phi}_x = \pm \mathrm{id}_M \qquad x \in (-\epsilon, \epsilon).$$

If we now set $T := \Theta^{-1}(\frac{\partial}{\partial x})$ on the set $\pi^{-1}[0, 1]$, then extend the definition in each direction using Φ and Φ^{-1}, T satisfies the conditions of theorem 15. □

Corollary 17. Let $A : \mathbb{R} \to \mathbb{R}$ be any unit periodic function. Then there is a symmetry vector field X for Φ such that $\pi_* X = A\frac{\partial}{\partial x}$.

Proof. We have from the theorem that there is always a symmetry T such that $\pi_* T = \frac{\partial}{\partial x}$. From proposition 9, $(\pi^* A)T$ is also a symmetry. □

Following from this result, if an OΔE defined by Φ has k independent evolutionary symmetry vector fields, there is necessarily an additional independent symmetry generating a trivialization in which Φ becomes autonomous. Thus we have the following theorems from corollary 17 and propositions 12 and 13.

Theorem 18. Let $\Phi : E \to E$ define a first-order OΔE which can be linearized on some open neighbourhood $U \subset E$. Then there exist $m + 1$ point-wise independent symmetry vector fields for Φ on U.

Theorem 19. Suppose that $\Phi : E \to E$ can be reduced to $\phi : F \to F$, as in definition 3. Then there exist locally $m - n + 1$ point-wise independent symmetry vector fields for Φ.

5.1. Example

Consider the system of two first-order OΔE defined in the octant $x, u, v > 0$ by

$$u_{x+1} = u_x^{v_x/(v_x+x)}$$
$$v_{x+1} = \frac{x+1}{x}(v_n + x).$$

Here $u_x \equiv u(x)$, etc. The corresponding map Φ is

$$\Phi(x, u, v) = \left(x + 1, u^{v/(v+x)}, \frac{x+1}{x}(v + x)\right)$$

with corresponding derivative map

$$[\Phi_*] = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{-v\ln(u)}{(v+x)^2}u^{v/(v+x)} & \dfrac{v}{v+x}u^{-x/(v+x)} & \dfrac{x\ln(u)}{(v+x)^2}u^{v/(v+x)} \\ 1 - \dfrac{v}{x^2} & 0 & 1 + \dfrac{1}{x} \end{bmatrix}.$$

If we look for a symmetry vector of the form $V_0 = \frac{\partial}{\partial x} + f(x, u, v)\frac{\partial}{\partial v}$, then the requirement $V_0 \circ \Phi = \Phi_* V_0$ is equivalent to the two conditions

$$xf(x, u, v) - v = 0$$

$$(1 + 1/x)f(x, u, v) + 1 - v/x^2 = f(x + 1, u^{v/(v+x)}, (1 + 1/x)(v + x)).$$

The first gives $f(x,u,v) = v/x$ and the second is consistant. Thus $V_0 = \frac{\partial}{\partial x} + \frac{v}{x}\frac{\partial}{\partial v}$ is a symmetry vector field with $\pi_* V_0 = \frac{\partial}{\partial x}$.

According to theorem 16, if we now choose the trivialization of $E = \mathbb{R}^3$ determined by V_0, then Φ will become autonomous.

Note that $V_0 u = 0$, so u is a first integral. Also $V_0 g(v/x) = 0$ for any smooth function g. Hence we take as new coordinates $\hat{u} := u$, $\hat{v} := v/x$ and $\hat{x} := x$. We then have $V_0 = \frac{\partial}{\partial \hat{x}}$ as expected and the OΔE takes the autonomous form

$$\hat{u}_{x+1} = \hat{u}_x^{\hat{v}_x/(\hat{v}_x+1)}$$

$$\hat{v}_{x+1} = v_n + 1.$$

6. Reduction of order

Having established in the previous section that symmetries are a consequence of linearization or reduction of an OΔE, we obviously wish to show that the converse also holds. This will be done by induction: first we must show that a single symmetry can be used to reduce the dimension of the system by one.

Lemma 20. Let $\Phi : E \to E$ define a first-order OΔE with a nowhere zero evolutionary symmetry vector field X. Then for some open submanifold $U \subseteq E$ there is a reduction of $\Phi : U \to U$ to $\phi : F \to F$, where F is a sub-bundle of E with co-dimension one.

Proof. The submanifold U can be chosen so that X generates a regular action ψ on U (see, e.g., [7]). Now F can be defined as the quotient manifold U/ψ, with μ mapping each point of U to the orbit of ψ on which it lies. Since X is evolutionary the fibre structure of U passes to the quotient and the orbits of ψ are one dimensional, so F is a sub-bundle of E with co-dimension one. Note that F need not be Hausdorff: Palais [8] has shown that this does not significantly restrict our use of F. Alternately, U may be chosen so that F is Hausdorff. □

To prove the induction step we must show that the remaining symmetries (of appropriate type) pass to the quotient.

Lemma 21. Let $\Phi : E \to E$ have a reduction to $\phi : F \to F$ given by $\mu : E \to F$ and suppose that the codimension of F is k. Let $X_1, \ldots, X_k$ span $\ker \mu_*$. Then if Y is a symmetry vector field of Φ on E which satisfies

$$[Y, X_j] \in \ker \mu_* \qquad j = 1, \ldots, k$$

there is a symmetry vector field $\tilde{Y}$ of ϕ on F.

Proof. The definition of the Lie bracket and the fact that $X_j \in \ker \mu_*$ implies that $\mu_* Y$ is well defined on the quotient, so we set $\tilde{Y} := \mu_* Y$. Thus we need to prove that $(\mu_* Y) \circ \phi = \phi_* \mu_* Y$. However, from the commutative diagram preceding definition 3 we have that $\phi_* \mu_* = \mu_* \Phi_*$. Moreover since Y is a symmetry of Φ, $\Phi_* Y = Y \circ \Phi$. Substituting on the right-hand side we now need to show only that $(\mu_* Y) \circ \phi = \mu_*(Y \circ \Phi)$, which follows from the same commutative diagram. □

We can now prove the theorem:

Theorem 22. Suppose an OΔE $\Phi : E \to E$ has s independent, evolutionary symmetry vector fields $X_1, \ldots, X_s$ which satisfy

$$[X_i, X_j] = \sum_{k=1}^{j-1} \lambda_{ij}^k X_k \qquad i < j \quad \lambda_{ij}^k \in C^\infty(E).$$

Then there is a reduction of Φ to $\phi : F \to F$ on some open submanifold $U \subseteq E$, with F having co-dimension s in E.

Proof. The proof is by induction. Suppose that we are able to construct a reduction $\mu_k : E \to F_k$, $0 \leqslant k \leqslant s-1$, with reduced O$\Delta$E $\phi_k : F_k \to F_k$. Here we identify $F_0 \equiv E$, $\phi_0 \equiv \Phi$ and $\mu_0 \equiv \mathrm{id}_E$. Suppose further that the vector field $\mu_{k*}X_{k+1}$ is well defined and generates a symmetry of ϕ_k on F_k, while $X_1, \ldots, X_k$ span $\ker \mu_{k*}$.

Now use the process of lemma 20 to define a reduction $\sigma : F_k \to F_{k+1}$, where F_{k+1} is the space of orbits of $\mu_{k*}X_{k+1}$ and σ is the canonical projection (with a restriction to an open submanifold of F_k if necessary to ensure regular orbits and a Hausdorff topology for F_{k+1}). Since $\mu_{k*}X_{k+1}$ spans $\ker \sigma_*$, defining $\mu_{k+1} := \sigma \circ \mu_k$ we have that $X_1, \ldots, X_{k+1}$ span $\ker \mu_{k+1*}$. Let ϕ_{k+1} be the reduction of Φ by μ_{k+1}.

Since

$$[X_{k+2}, X_j] \in \ker \mu_{k*} \qquad j = 1, \ldots, k+1$$

we know from lemma 21 that $\mu_{k+1*}X_{k+2}$ is a well-defined symmetry vector field of $\phi_{k+1} : F_{k+1} \to F_{k+1}$ and so the induction is complete. The $k = 0$ case follows immediately from lemma 20. □

Of course, reduction of order is only half the story. It allows the construction of a complicated difference equation starting from a simple one, so that the solutions of the simple system pass to those of the complicated system by the quotienting procedure, however, the technique was originated for ODEs in order to solve the larger equation from the solution of the reduced system. To do this we must have a method of reconstructing solutions, or ideally a linearization, of the larger system from solutions or linearization of the smaller.

Theorem 23. Let $\Phi : E \to E$ define an OΔE and suppose $\mu : E \to F$ is a reduction to $\phi : F \to F$ by the action of an integrable system of symmetry vector fields as in theorem 22. Let $x, u^1, \ldots, u^{m-s}$ be coordinates on F. Then we can construct coordinates $x, u^1, \ldots, u^{m-s}, v^1, \ldots, v^s$ for E such that

$$\begin{aligned} &\Phi^0(x, u, v) = x + 1 \\ &\Phi^j(x, u, v) = \phi^j(x, u) \qquad j = 1, \ldots, m-s \\ &\Phi^k(x, u, v) = v^{k-m+s} \qquad k = m-s+1, \ldots, v^s. \end{aligned}$$

In particular, if ϕ is completely linearized then Φ will be completely linearized.

Proof. We will prove the result by induction. Let F_k be one of the intermediate reductions $\mu_k : E \to F_k$, as in the proof of theorem 22. Assume $F_s = F$. We have coordinates $(x, u^1, \ldots, u^{m-k}) \equiv (x, u)$ on F_k with respect to which

$$\phi_k^j(x, u) = f_k^j(x, u) \qquad j = 1, \ldots, m-k.$$

Let $\sigma : F_{k-1} \to F_k$ be the reduction generated by X_k. The coordinate functions (x, u) pull back by σ to define $m-k+1$ independent coordinates on F_{k-1}, which we will also call $(x, u^1, \ldots, u^{k-1})$. To complete the local chart on F_{k-1}, we can choose any function v which is everywhere independent of those pulled back from F_s. Since $x, u^1, \ldots, u^{m-s}$ are invariants of the action of X we must have $\frac{\partial}{\partial v} \in \ker \sigma_*$.

Since $\ker \sigma_*$ is one-dimensional, this implies $X_k = \eta(x, u, v)\frac{\partial}{\partial v}$ for some function η.

Now from the definition of a reduction, $\sigma \circ \phi_{k-1} = \phi_k \circ \sigma$, so

$$\phi^j_{k-1}(x, u, v) = (\sigma \circ \phi_{k-1})^j(x, u, v) = \phi^j_k(x, u) = f^j_k(x, u) \qquad j = 1, \ldots, m-k.$$

The remanining component of ϕ_{k-1} is now a one-dimensional parametrized OΔE

$$v(x+1) = \phi^{m-k+1}_{k-1}(x, u(x), v(x)). \tag{2}$$

Moreover, since X_k generates a symmetry of ϕ_{k-1} it is also a symmetry of the $m-k+1$th component. We will use this to linearize (2).

Maeda [6] showed how to linearize a single autonomous OΔE given an autonomous symmetry vector field. We now show that his method can be generalized to the time dependent, parametrized case.

Define new coordinates (treating the u^j as parameters)

$$\tilde{v} := \int^{\tilde{v}} \frac{\mathrm{d}v}{\eta(x, u, v)}$$
$$\tilde{x} := x.$$

It follows that $\frac{\partial}{\partial \tilde{v}} = \eta\frac{\partial}{\partial v}$, so that $X = \frac{\partial}{\partial \tilde{v}}$. Since X is still a symmetry of ϕ^{m-k+1}_{k-1} in these coordinates,

$$\phi^{m-k+1}_{k-1*}X = \frac{\partial\phi^{m-k+1}_{k-1}}{\partial\tilde{v}}\frac{\partial}{\partial\tilde{v}} = X \circ \phi^{m-k+1}_{k-1} = \frac{\partial}{\partial\tilde{v}}$$

so that $\phi^{m-k+1}_{k-1}(x, u, \tilde{v}) = (x+1, u, \tilde{v} + g(x, u))$.

If g depends only on the coordinates $u^{m-s+1}, \ldots, u^{m-k}$, which are necessarily unit periodic, we can immediately define $u^{m-k+1}(x, u, \tilde{v}) := \tilde{v} - xg(u)$. Setting

$$f^j_{k-1} = f^j_k \qquad j = 1, \ldots, m-k$$
$$f^{m-k+1}_{k-1}(x, u^1, \ldots, u^{m-k+1}) = u^{m-k+1}$$

then decrementing k to $k-1$, we continue by induction (starting from $k = s$) until the theorem is proved.

In the general case we need to do some more work. Let χ be any smooth function such that $\chi(x) = 0$ for all x in a neighbourhood of 0 and $\chi(x) = 1$ for all x in a neighbourhood of 1. Then with the notation $[x]$ denoting the largest integer less than or equal to x, we set

$$u^{m-k+1}_x := \tilde{v}_x - \chi(x-[x])g(x-[x], u_{x-[x]}) - \sum_{j=1}^{[x-1]} g(x-j, u_{x-j}). \tag{3}$$

It can be shown that u^{m-k+1} is a smooth function of $(x, u, \tilde{v})$ and that (2) is linearized with respect to this coordinate. We then use induction as in the autonomous case above. $\square$

It is worth remarking here that the vector fields X_k must be evolutionary for this process to work: the behaviour of $\frac{\partial}{\partial x}$ under the coordinate transformations used above is very awkward.

6.1. Example 1

Recall the example of section 5.1:

$$u_{x+1} = u_x^{v_x/(v_x+x)}$$

$$v_{x+1} = \frac{x+1}{x}(v_x + x).$$

Apart from the time-like symmetry $V_0 = \frac{\partial}{\partial x} + (v/x)\frac{\partial}{\partial v}$ given previously, there are two further symmetries:

$$V_1 = \frac{xu}{v}\frac{\partial}{\partial u}$$

$$V_2 = -\frac{xu}{v}\ln(u)\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}.$$

These are both evolutionary vector fields. Moreover $[V_1, V_2] = 0$, so the order in which they are used to reduce the system is unimportant.

As V_1 is simpler, we use it first. The corresponding projection μ is given by

$$\mu : (x, u, v) \mapsto (x, v).$$

The reduced OΔE ϕ is given by $\phi(x, v) = \mu \circ \Phi(x, u, v)$, so

$$\phi(x, v) = \left(x + 1, \left(1 + \frac{1}{x}\right)(v + x)\right).$$

Now since $[V_1, V_2] = 0$, $\mu_*(V_2) = x\frac{\partial}{\partial v}$ is a symmetry vector field of ϕ. Thus we define a coordinate

$$\tilde{v} := \frac{v}{x} = \int^{\tilde{v}} \frac{\mathrm{d}v}{x}.$$

Using this we have $\tilde{v}_{x+1} = \tilde{v}_x + 1$, so a further change to $\hat{v}_x := \tilde{v}_x - x = v/x - x$ completely linearizes the reduced system:

$$\hat{v}_{x+1} = \hat{v}_x.$$

Reconstructing,

$$\Phi(x, u, \hat{v}) = (x + 1, u^{(\hat{v}+x)/(\hat{v}+x+1)}, \hat{v})$$

which retains the symmetry

$$V_1 = \frac{u}{\hat{v} + x}\frac{\partial}{\partial u}.$$

Re-writing in terms of the new variable

$$\hat{u} := (\hat{v} + x)\ln(u) = \int^{\hat{u}} \frac{\mathrm{d}u}{u/(\hat{v} + x)}$$

we find $\hat{u}_{x+1} = \hat{u}_x$. Thus the system is completely linearized in the coordinates $(x, \hat{u}, \hat{v})$.

Solving the OΔE in the original coordinates is now just a matter of expressing u and v in terms of $\hat{u}$ and $\hat{v}$. These 'first integrals' can then be replaced by arbitrary unit-periodic functions (provided the condition $x, u, v > 0$ is observed):

$$u(x) = \exp\left(\frac{\hat{u}(x)}{\hat{v}(x) + x}\right)$$

$$v(x) = x(\hat{v}(x) + x)$$

where $\hat{u}(x + 1) = \hat{u}(x)$, $\hat{v}(x + 1) = \hat{v}(x)$.

6.2. *Example 2*

Now consider an example which leads to a more complicated reconstruction using the substitution (3):

$$u_{x+1} = xu_x$$

$$v_{x+1} = u_x v_x.$$

The corresponding mapping is

$$\Phi(x, u, v) = (x + 1, xu, uv)$$

with derivative map in matrix form

$$[\Phi_*] = \begin{bmatrix} 1 & 0 & 0 \\ u & x & 0 \\ 0 & v & u \end{bmatrix}.$$

The two evolutionary symmetries

$$X_1 := v\frac{\partial}{\partial v} \qquad X_2 := u\frac{\partial}{\partial u} + xv\frac{\partial}{\partial v}$$

can then be found and the commutator $[X_1, X_2] = 0$ computed.

Since u is an invariant of X_1, define the projection

$$\mu : (x, u, v) \mapsto (x, u).$$

The reduced equation comes from the map

$$\mu \circ \Phi(x, u, v) = \phi(x, u) = (x + 1, xu)$$

and must admit the projected vector field

$$\mu_* X_2 = u\frac{\partial}{\partial u}$$

as a symmetry. Thus we define the new coordinate

$$\tilde{u} := \int^{\tilde{u}} \frac{du}{u} = \ln u.$$

In the new coordinates the reduced equation has the form

$$\tilde{u}_{x+1} = \tilde{u}_x + \ln x.$$

Following equation (3) we now define

$$\hat{u}_x := \tilde{u}_x - \chi(x - [x])\ln(x - [x]) - \sum_{j=1}^{[x-1]} \ln(x - j)$$

where χ is a smooth step function as described above. Thus we have $\hat{u}_{x+1} = \hat{u}_x$.

Reconstructing, the original pair of equations becomes

$$\hat{u}_{x+1} = \hat{u}_x$$

$$v_{x+1} = u(x, \hat{u}) v_x$$

and the second equation admits the symmetry vector field X_1. Again we define

$$\tilde{v} := \int^{\tilde{v}} \frac{\mathrm{d}v}{v} = \ln v$$

leading to

$$\tilde{v}_{x+1} = \tilde{v}_x + \hat{u}_x + \chi(x - [x]) \ln(x - [x]) + \sum_{j=1}^{[x-1]} \ln(x - j).$$

Using equation (3) again and re-arranging summations we obtain

$$\hat{v}_x = \ln v_x - x\hat{u}_x - \{\chi(x - [x]) + [x - 1]\} \chi(x - [x]) \ln(x - [x])$$
$$- \sum_{j=2}^{[x-1]} (j - 2) \ln(x - j).$$

The term $-x\hat{u}_x$ has been obtained by using the fact that $\hat{u}$ is a unit-periodic function.

In the coordinates $(x, \hat{u}, \hat{v})$ the system is completely linearized and can be solved by replacing $\hat{u}$ and $\hat{v}$ with arbitrary unit-periodic functions of x. Finding the general solution of the original system is then a matter of expressing u, v in terms of $x, \hat{u}, \hat{v}$:

$$u(x) = \mathrm{e}^{\hat{u}(x)} \left(\prod_{j=1}^{[x-1]} (x - j) \right) (x - [x])^{\chi(x-[x])}$$

$$v(x) = \mathrm{e}^{\hat{v}(x) + x\hat{u}(x)} \left(\prod_{j=2}^{[x-1]} (x - j)^{j-1} \right) \ln(x - [x])^{(\chi(x-[x])+[x-1])\chi(x-[x])}.$$

7. Summary and concluding remarks

As with differential equations, the presence of a sufficient number of symmetries forming an integrable system implies complete linearization via reduction. Such symmetries always exist for systems of non-singular first-order ordinary differential equations. However, linearization of OΔEs is possible if and only if the defining map Φ is orientation preserving.

In fact we have shown that if Φ preserves orientation, then Φ is the unit-time flow of a vector field. Consequently it is the unit-time flow of an equivalence class of vector fields.

A symmetry of an OΔE is therefore a simultaneous symmetry of an equivalence class of ODEs. It is this requirement of simultaneously preserving solutions of a class of ODEs which imposes a restriction on symmetries of OΔEs: they must be projectable.

Once the symmetries of a system of OΔEs are known, the process of reduction and reconstruction is essentially the same as for ODEs: it is necessary to find the invariants of the symmetries by integrating a Pfaffian system.

Finding the symmetries is slightly different. The determining equations for the symmetries of ODEs are linear, first-order PDEs for the coefficients of the symmetry vector fields. Those for systems of OΔEs are linear, first-order partial *difference* equations. As with ODEs, the determining equations may be more difficult to solve than the original problem. However, in the autonomous case Gaeta [4] has shown that the symmetries may be determined from a formal series expansion, similar to the Poincaré–Dulac procedure [1]. It should be possible to extend this technique to the time dependent case: certainly this is so if we can first determine a time-like symmetry and reduce to an autonomous system using theorem 15. The convergence of such a series needs further study.

In the cases where the symmetries can be determined from geometric arguments or other fortuitous circumstances, the theory presented in this paper is an important tool for the simplification or solution of difference equations.

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